A Theorem on the Zero Divisor Graph of the Ring of 2×2 - matrices over $\mathbb{Z}_2$

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ABSTRACT

In this paper we consider the ring R of 2×2 - matrices over $\mathbb{Z}_2$ (the ring of integers modulo 2). We constructed ZDG(R) (the zero divisor graph of the ring R) and obtained a simple fundamental important result on ZDG(R). Finally we observed some basic properties of ZDG(R).

Keywords: Zero divisor graph, Star graph, Associative ring.

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1 Introduction

Let G = (V, E) be a graph consist of a finite non-empty set V of vertices and finite set E of edges such that each edge e_k is identified as an unordered pair of vertices $\{v_i, v_j\}$, where v_i, v_j are called end points of e_k. The edge e_k is also denoted by either $v_i v_j$ or $\overline{v_i v_j}$. We also write $G(V, E)$ for the graph. Vertex set and edge set of G are also denoted by $V(G)$ and $E(G)$ respectively. An edge associated with a vertex pair $\{v_i, v_i\}$ is called a self-loop. The number of edges associated with the vertex v is called as the degree of the vertex, and it is denoted by d(v). If there is more than one edge associated with a given pair of vertices, then these edges are called parallel edges or multiple edges. A graph that does not have self-loops or parallel edges is called a simple graph. We consider simple graphs only. A vertex of degree 1 is called as pendent vertex.

1.1 Definitions: (i) A graph G (V, E) is said to be a star graph if there exists a fixed vertex v such that E = {vu / u $\in$ V and u $\neq$ v}. A star graph is said to be an n-star graph if the number of vertices of the graph is n.

(ii) In a graph G, a subset S of V (G) is said to be a *dominating set* if every vertex not in S has a neighbor in S. The *domination number*, denoted by γ(G) is defined as min {|S| / S is a dominating set in G}

(iii) In a connected graph, a closed walk running through every edge of G exactly once (except the starting vertex at which the walk terminates) is called as **Hamiltonian circuit.** A graph containing a Hamiltonian circuit is called as **Hamiltonian graph.**

1.2 Theorem: (Th. 13.8, page 361, [4]) A given connected graph G is an Eulerian graph if and only if all the vertices of G are of even degree.

For other fundamental concepts we refer [2], [3], [4] or [5]

2. Zero divisor graph of a ring

Throughout R denotes a finite associative ring need not be commutative.

2.1 Definition: A graph $G=(V,E)$ is said to be the zero divisor graph of R if $V=R$ and

E = $\left\{ \overline{xy} / x \neq y,\ x,y \in R,\ x \neq 0 \neq y,\ xy=0 \right\} \cup \left\{ \overline{x0} \ / 0 \neq x \in R \right\}$ where $\overline{xy}$

denotes an edge between $x, y \in V$.

2.2 Note: (i) This definition 'zero divisor graph' is same as that of Beck [1988] in case of commutative rings.

(ii) we denote the zero divisor graph of a ring R by ZDG(R). It is clear that $V(ZDG(R)) = R$ and

$E(ZDG(R))$ = $\left\{ \overline{xy} / x \neq y,\ x,y \in R,\ x \neq 0 \neq y,\ xy=0 \right\} \cup \left\{ \overline{x0} \ / 0 \neq x \in R \right\}$.

2.3 Example: Consider $\mathbb{Z}_2 = \{0,1\}$ the ring of integers modulo 2.

Write R = Set of all 2×2 matrices with the elements from $\mathbb{Z}_2$. Note that R is a ring which is not commutative. Now let us construct ZDG(R) in the following. We use the notation given below.

$V(ZDG(R))$ = $\{v_i / 0 \leq i \leq 15\}$ *where*

$$v_0=\begin{pmatrix}0&0\\0&0\end{pmatrix}, v_1=\begin{pmatrix}0&0\\0&1\end{pmatrix}, v_2=\begin{pmatrix}0&0\\1&0\end{pmatrix}, v_3=\begin{pmatrix}0&0\\1&1\end{pmatrix}, v_4=\begin{pmatrix}0&1\\0&0\end{pmatrix}, v_5=\begin{pmatrix}0&1\\0&1\end{pmatrix},$$

$$v_6=\begin{pmatrix}0&1\\1&0\end{pmatrix}, v_7=\begin{pmatrix}0&1\\1&1\end{pmatrix}, v_8=\begin{pmatrix}1&0\\0&0\end{pmatrix}, v_9=\begin{pmatrix}1&0\\0&1\end{pmatrix}, v_{10}=\begin{pmatrix}1&0\\1&0\end{pmatrix}, v_{11}=\begin{pmatrix}1&0\\1&1\end{pmatrix},$$

$$v_{12}=\begin{pmatrix}1&1\\0&0\end{pmatrix}, v_{13}=\begin{pmatrix}1&1\\0&1\end{pmatrix}, v_{14}=\begin{pmatrix}1&1\\1&0\end{pmatrix}, v_{15}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$$

We note that $v_1v_2 = v_2 \neq v_0 = v_2v_1$

Since $v_0v_i = 0$ for all $1 \leq i \leq 15$ we have that v_0 is adjacent to all other vertices of ZDG(R).

Also we observe that

$$v_2v_1 = v_1v_4 = v_1v_8 = v_1v_{12} = v_2v_3 = v_3v_5 = v_3v_{10} = v_3v_{15} = v_4v_8 = v_4v_{12} = v_5v_8 = v_5v_{12} = v_{12}v_{15} = v_8v_1 = v_8v_2$$
$$= v_8v_3 = v_{10}v_1 = v_{10}v_2 = v_{10}v_3 = v_{12}v_{10} = v_{12}v_{15} = v_{15}v_5 = v_5v_4 = 0$$

So $E(ZDG(R))$ =

$$\left\{\begin{array}{l}\overline{v_0v_1}, \overline{v_0v_2}, \overline{v_0v_3}, \overline{v_0v_4}, \overline{v_0v_5}, \overline{v_0v_6}, \overline{v_0v_7}, \overline{v_0v_8}, \overline{v_0v_9}, \overline{v_0v_{10}}, \overline{v_0v_{11}}, \overline{v_0v_{12}}, \overline{v_0v_{13}}, \overline{v_0v_{14}}, \overline{v_0v_{15}}, \overline{v_1v_2}, \overline{v_1v_4}, \overline{v_1v_8}, \\ \overline{v_1v_{12}}, \overline{v_2v_3}, \overline{v_3v_5}, \overline{v_3v_{10}}, \overline{v_3v_{15}}, \overline{v_4v_8}, \overline{v_4v_{12}}, \overline{v_5v_8}, \overline{v_5v_{12}}, \overline{v_{12}v_{15}}, \overline{v_8v_1}, \overline{v_8v_2}, \overline{v_8v_3}, \overline{v_{10}v_1}, \overline{v_{12}v_{10}}, \overline{v_{15}v_5}, \overline{v_5v_4}\end{array}\right\}$$

The zero divisor graph of R, that is ZDG(R) is given by the Fig. 2.3.

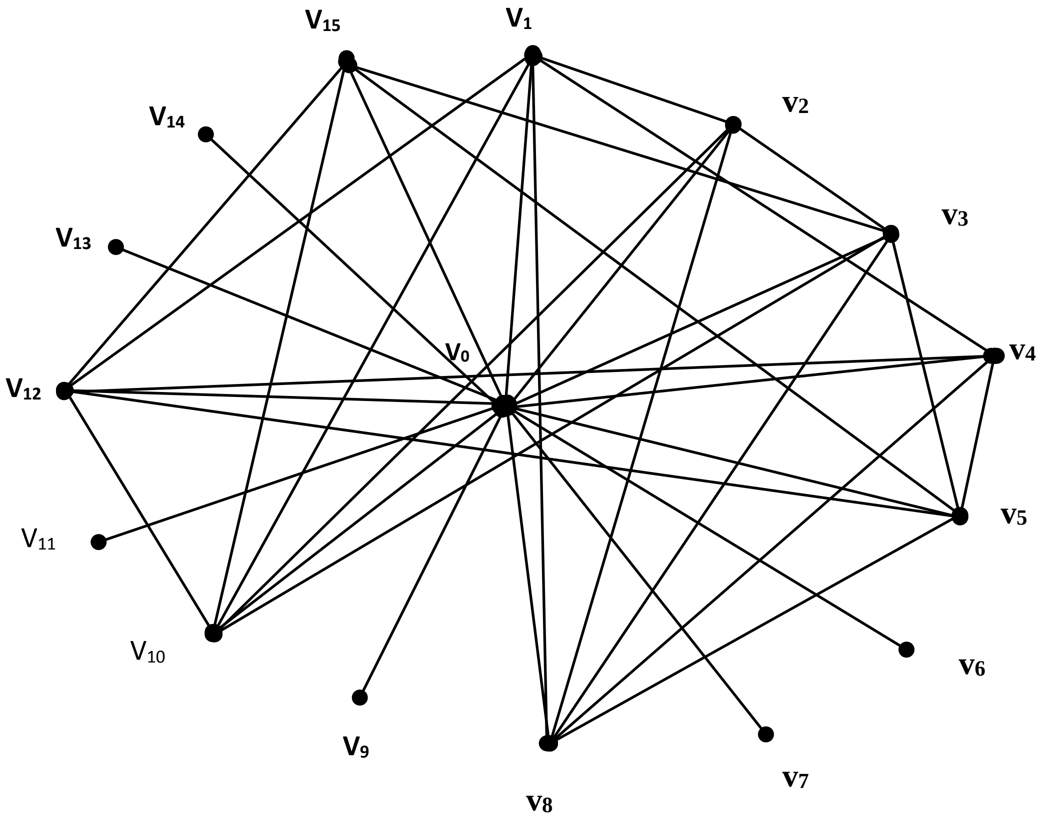

Fig. 2.3

Now we prove the following fundamental theorem.

2.4 Theorem: The ZDG(R) where R is the set of all 2×2 - matrices over $\mathbb{Z}_2$ contains six pendent vertices.

Proof: we got this in five simple parts

Part (i): Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$ such that $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$.

Then $\begin{pmatrix} c & d \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ c = d = 0, a + c = 0, b + d = 0

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$. So $d(v_7) = 1$.

In the same way, one can observe that $\mathrm{d}(\mathrm{v}_{11}) = \mathrm{d}(\mathrm{v}_{13}) = \mathrm{d}(\mathrm{v}_{14}) = 1$.

Part (ii): Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R$ such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$

Then $\begin{pmatrix} c & d \\ a & b \end{pmatrix} = 0 \Rightarrow a = b = c = d = 0$. So $d(v_6) = 1$.

Similarly one can verify that $d(v_9) = 1$.

Part (iii): Observe that $\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$ and so $d\begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \geq 2$ for $x, y \in \mathbb{Z}_2$.

So $d(v_0), d(v_1), d(v_4), d(v_5) \notin \{0,1\}$..

Part (iv): Since $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = 0$ we have that $d(v_2), d(v_8), d(v_{10}) \notin \{0,1\}$

Part (v): $v_3v_{10} = v_{12}v_{10} = v_{15}v_5 = 0$ we have that $d(v_3), d(v_{12}), d(v_{15}) \notin \{0,1\}$. Therefore the set of all pendent vertices is $\{v_6, v_7, v_9, v_{11}, v_{13}, v_{14}\}$.

Finally, we list out the following properties of the graph ZDG(R).

2.5. Some Properties of ZDG(R) (refer Fig 2.3).

(i) Since v_0 dominates the graph, the domination number of ZDG(R) is 1

(ii) Since the graph ZDG(R) contains cycles, it cannot be a bipartite graph

(iii) Since $v_0v_i = 0$ for all $1 \leq i \leq 15$ the graph ZDG(R) contains a 16 – star graph.

(iv) The graph ZDG(R) contains exactly six pendent vertices (by Theorem 2.4)

(v) ZDG(R) is not an Eulerian graph (by using the Th. 13.8, p 361 of [4]).

(vi) Since ZDG(R) contains pendent vertices, it contains no Hamiltonian circuit.

References

[1]. Beck, Coloring of Commutative Ring, J. Algebra 116 (1988), 208-226.

[2]. Satyanarayana Bh. and Syam Prasad K. "An Isomorphism Theorem on Directed Hypercubes of Dimension n", *Indian J. Pure & Appl. Math* 34 (10) (2003) 1453-1457.

[3]. Satyanarayana Bh. and Syam Prasad K. "Discrete Mathematics and Graph Theory", Prentice Hall India Pvt. Ltd., New Delhi, (2009) (ISBN 978-81-203-3842-5).

[4]. Satyanarayana Bh. and Syam Prasad K. "Discrete Mathematics and Graph Theory", Prentice Hall India Pvt. Ltd., New Delhi, 2014 (Second Edition) (ISBN 978-81-203-4948-3).

[5]. Satyanarayana Bh. and Syam Prasad K. "Nearrings, Fuzzy Ideals and Graph Theory" CRC Press (Taylor & Francis Group, London, New York), (2013) (ISBN 13: 9781439873106).

[6]. Satyanarayana Bhavanari., Syam Prasad K and Nagaraju D. "Prime Graph of a Ring", J. Combinatorics, Informations & System Sciences 35 (2010) 27-42.

[7]. Satyanarayana Bhavanari., Srinivasulu D. "Cartesian Product of Graphs Vs. Prime Graphs of Rings", Global Journal of Pure and Applied Mathematics(GJPAM), Volume 11,Number 2 (2015). (ISSN 0973-1768).

[8]. Satyanarayana Bhavanari , Srinivasulu D., and Mallikarjun Bhavanari "A Theorem on the Prime Graph of 2×2 - matrix ring of $\mathbb{Z}_2$", (Communicated).

[9]. Satyanarayana Bhavanari., Srinivasulu D., and Syam Prasad K. "Some Results on Degree of Vertices in Semitotal Block Graph and Total – Block Graph", International Journal of computer Applications, Vol. 50, No.9, (July 2012) 19-22. (ISSN: 0975 - 8887).

[10]. Satyanarayana Bhavanari., Srinivasulu D., Syam Prasad K. "Line Graphs& Quasi-Total Block Graphs", International Journal of Computer Applications Vol. 105, No.3, (November 2014) 12-16. (ISSN: 0975 - 8887).

[11]. Satyanarayana Bhavanari., Pradeep kumar T.V, Sk. Mohiddin Shaw. "Mathematical Foundations of Computer Science", BS publications, Hyderabad, A.P, India, 2016. (ISBN: 978-93-83635-81-8).