THE GENERALIZED RANDERS CHANGE WITH MTH ROOT METRICS OF A FINSLER SPACE INCLUDING SOME OF ITS INTRINSIC PROPERTIES

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ABSTRACT

The purpose of present paper is devoted to studying a condition under which a Finsler space with Randers change of mth root is projectively related to a mth root metric. Further we have studied the conditions under which generalized Randers metric reduces to Rander metric and types of Finsler spaces arising from this metric.

KEYWORDS: Finsler space, Randers space, mth root metric.

1. Introduction

A change of Finsler metric $F(x,y) \to \overline{F}(x,y)$ is called a generalized Randers change of F, if

$$\overline{F}(x,y) = F(x,y) + b_i(x,y)y^i$$

where$b_i(x,y)$ is h-vector in n-dimensional Finsler manifold (M,F). If a differential 1-form $\beta(x,y) = b_i(x)y^i$ is given on M^n ,**M,Matsumoto[10]** introduced another Finsler space whose fundamental function is given by

$$\overline{F}(x,y) = F(x,y) + \beta(x,y)$$

This change of Finsler metric has been called β-change **[1,3].**

The m^{th} root metric was firstly introduced by **Shimada [6]**in 1979.

The second root metric is a Riemannian metric, so it is regarded as a generalization of Riemannian metric. Recent studies show that m^{th} rootFinsler metrics play a very important role in physics, space time and general relativity as well as in unified gauge field theory.**See ([4][14][12][15][5])**

Recently **S.H.Abed [11]**have generalized the metric with the help of h- vector and have introduced another Finsler metric defined as

$$\overline{F}(x,y) = e^{\sigma(x)}F(x,y) + \beta(x,y)$$

where$\sigma(x)$ is a function of x and $\beta(x,y) = b_i(x,y)y^i$ is a 1-form on M^n and b_i satisfies the condition of being an h-vector. We call the change $F(x,y) \to \overline{F}(x,y)$ as h-Randers conformal change. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a h- Randers change **[2].** When $\beta = 0$ and σ is a non-zero constant then it reduces to a homothetic change. It reduces to Randers conformal change when b_i and σ are functions of position only.**[11,8]**

1. The m-th root Metric

In 1941 ,**Randers [6**] introduced a Finsler metric $F = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannain metric and $\beta = b_i(x)y^i$ is a differential one- form. In 1971 a Finsler metric

$$\bar{F}(x,y) = F(x,y) + \beta(x,y)$$

where F is a Finsler metric and β is a one-form on the manifold M^n , was introduced by M.Matsumoto. This metric is called Randers change of Finsler metric.

Consider the transformation

$$\bar{F} = F + \beta \qquad (1.1)$$

Where $F = \sqrt[m]{A}$ is an m-th root metric and $\beta(x,y) = b_i(x)y^i$ is a one-form on the manifold M^n.

The differentiation of (1.1) with respect to y^i yields the normalized supporting element $\bar{l}_i$ given by

$$\bar{l}_i = l_i + b_i$$

$$\bar{l}_i = \frac{A_i}{F^{m-1}} + b_i \text{ as } l_i = \frac{A_i}{F^{m-1}} \qquad (1.2)$$

Again differentiating (1.2) with respect to y^j yields:

$$\bar{h}_{ij} = (m-1)\frac{\bar{F}}{F}\left(\frac{A_{ij}}{F^{m-2}} - \frac{A_i A_j}{F^{2(m-1)}}\right) \qquad (1.3)$$

From (1.2) and (1.3) the fundamental metric tensor $\bar{g}_{ij}$ of Finsler space$\bar{F}^n$ is given by

$$\bar{g}_{ij} = \bar{h}_{ij} + \bar{l}_i \bar{l}_j$$

$$\bar{g}_{ij} = (m-1)\frac{\bar{F}}{F}\left(\frac{A_{ij}}{F^{m-2}} - \frac{A_i A_j}{F^{2(m-1)}}\right) + \left(\frac{A_i}{F^{m-1}} + b_i\right)\left(\frac{A_j}{F^{m-1}} + b_j\right) \qquad (1.4)$$

$$\bar{g}_{ij} = \frac{(m-1)\tau A_{ij}}{F^{m-2}} + \frac{A_i b_j + A_j b_i}{F^{m-1}} + b_i b_j + (1 - \overline{m-1}\tau)\frac{A_i A_j}{F^{2(m-1)}}$$

where $\tau = \frac{\bar{F}}{F}$

The contravariant metric tensor $\bar{g}_{ij}$ of Finsler space $\bar{F}^n$ is given by

$$\bar{g}^{ij} = \frac{1}{\tau(m-1)}\left\{F^{m-2}A^{ij} - \frac{1}{1+q}\frac{b^i y^j + b^j y^i}{F} + \frac{b^2 + \overline{(m-1)}\tau - 1}{(1+q)^2}\frac{y^i y^j}{F^2}\right\}$$

Thus the contravariant metric tensor $\bar{g}^{ij}$ is rewritten in the form

$$\bar{g}^{ij} = \frac{1}{\tau(m-1)}\left\{F^{m-2}A^{ij} - \frac{b^i y^j + b^j y^i}{\bar{F}} + \frac{b^2 + \overline{(m-1)}\tau - 1}{\bar{F}^2} y^i y^j\right\} \qquad (1.5)$$

Where $q = \frac{\beta}{F}$, $(1+q) = \frac{\bar{F}}{F} = \tau$ and the matrix (A^{ij}) denotes inverse of (A_{ij})**[17].** Here we have used $A^{ij}A_j = A^i = y^i$

2.Randers change

In 1979, **Shimada[7]** introduced the m-th root metric on the differentiable manifold M defined as

$$F = \sqrt[m]{a_{i_1 i_2 i_3 \ldots\ldots i_m}(x) y^{i_1} y^{i_2} \ldots . y^{i_m}}$$

where the coefficients $a_{i_1 i_2 i_3 \dots\dots i_m}$ are the components of symmetric covariant tensor field of order (0,m) being the functions of positional co-ordinates only.

There exist the following important two classes of Finsler metrics,

$$\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$$

$$\tilde{F} = \sqrt{A^{\frac{2}{m}} + B + C} \qquad (2.1)$$

Where $A = a_{i_1 i_2 i_3 \dots\dots i_m}(x) y^{i_1} y^{i_2} \dots . y^{i_m}$, $B = b_{ij}(x) y^i y^j$ and $C = c_k(x) y^k$, that is 1-form. These forms are called a generalized m-th root metric and more general generalized m-th root metric, respectively.

In [**14**], the geometric properties of locally projectively flat m-root in the form $F = \sqrt[m]{A}$ and generalized m-th root in the form

$$\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$$

Now we have considered a transformation of the more generalized m-th root metric such that it transforms to a similar metric as the generalized Randers

$$\bar{F}(x, y) = F(x, y) + b_i(x, y) y^i$$

In a way that the Finslerian metric F is replaced with more generalized m-th root metric $\tilde{F}$ defined in (2.1).

Then, we obtain the conditions among two more generalized m-th root metrics $\tilde{F}_1 = \sqrt{A_1{}^{\frac{2}{m_1}} + B_1 + C_1}$ and $\tilde{F}_2 = \sqrt{A_2{}^{\frac{2}{m_2}} + B_2 + C_2}$ due to generalized Randers change, when m_1, m_2 are even numbers. We prove under these conditions generalized Randers metric reduces to Randers metric.

Theorem 1. *Let* $\tilde{F}_1 = \sqrt{A_1{}^{\frac{2}{m_1}} + B_1 + C_1}$ *and* $\tilde{F}_2 = \sqrt{A_2{}^{\frac{2}{m_2}} + B_2 + C_2}$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$. *Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2$ *and* $m_1, m_2 > 2$. *If* $\tilde{F}_1$ *is generalized Randers change of* $\tilde{F}_2$, *then* $\tilde{F}_1$ *reduces to a Randers* β*- change of* $\tilde{F}_2$.

Proof. Suppose that $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$. Then

$$C_1 = C_2 + b_i(x, y) y^i \qquad (2.2)$$

Differentiating (2.2) with respect to y^k we have

$$C_k(x) = \bar{C}_k(x) + b_k(x, y) \qquad (2.3)$$

Then $\dot{\partial}_j b_k(x, y) = 0 \qquad (2.4)$

Therefore b_i are functions of coordinates x^i alone and b_i is not a h- vector.

3.C-reducibility of $\bar{F}^n$

Following Matsumoto **[10]**, in this section we shall investigate special cases of the Finsler space with h-Randers conformally changes Finsler space $\bar{F}^n$.

Definition 3.1 A Finsler space (M^n, L) with dimension $n \geq 3$ is said to be quasi-C-reducible if the Cartan tensor C_{ijk} satisfies

$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j$$

Where Q_{ij} is a symmetric indicatory tensor.

We have **[8]**

$$\bar{C}^h_{ij} = C^h_{ij} + \frac{1}{2\bar{L}}\left(h_{ij}m^h + h^h_j m_i + h^h_i m_j\right) - \frac{1}{L}\left[\left\{\rho + \frac{L}{2\bar{L}}(b^2 - \frac{\beta^2}{L^2})\right\} h_{ij} + \frac{L}{\bar{L}} m_i m_j\right] l^h \quad (3.1)$$

Substituting $h = j$ in (3.1) we get

$$\bar{C}_i = C_i + \frac{(n+1)}{2\bar{L}} m_i \qquad (3.2)$$

Using **[8]** and (3.2) we have

$$\bar{C}_{ijk} = \varphi C_{ijk} + \frac{\varphi}{(n+1)} \pi_{(ijk)}\{h_{ij}(\bar{C}_k - C_k)\}$$

Where $\pi_{(ijk)}$ represents cyclic permutation and sum over the indices $i, j \ and \ k$

The above equation can be written as

$$\bar{C}_{ijk} = \varphi C_{ijk} + \frac{\varphi}{(n+1)} \pi_{(ijk)}(h_{ij}\bar{C}_k) - \frac{\varphi}{(n+1)} \pi_{(ijk)}(h_{ij}C_k)$$

Definition 3.2 A Finsler space (M^n, L) of dimension $n \geq 3$ is called C-reducible if the Cartan tensor C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{(n+1)}\left(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i\right) \qquad (3.3)$$

Now from**[8]** and definition of C-reducibility we have

$$\varphi C_{ijk} = \pi_{(ijk)}\left(\bar{h}_{iij}N_k\right) \qquad (3.4)$$

Where $N_k = \frac{1}{(n+1)}\bar{C}_k - \frac{1}{2\tau} m_k$

Conversely, if (3.4) is satisfied or certain vector N_k then we have

$$\bar{C}_{ijk} = \frac{1}{(n+1)} \pi_{(ijk)}\left(\bar{h}_{ij}\bar{C}_k\right)$$

Thus we have

Theorem 3.1. *An h-Randers conformally changed Finsler Space $\bar{F}^n$ is C-reducible iff equation (3.4) holds good.*

Corollary 3.1. If the Finsler space F^n is C-reducible Finsler space, then an h-Randers conformally changed Finsler space $\bar{F}^n$ is always a C-reducible Finsler space.

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